Neumann-like integrable models

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Abstract

A countable class of integrable dynamical systems, with four dimensional phase space and conserved quantities in involution (H_n, I_n) are exhibited. For $n = 1$ we recover Neumann sytem on $T^* S^2$. All these systems are also integrable at the quantum level.

1 Introduction

The classical problem of motion of a rigid body in an ideal fluid leads to one among the oldest integrable models : Clebsch dynamical system [1]. Upon symplectic reduction it becomes Neumann celebrated integrable system [3] with phase space T^*S^2. Its Hamiltonian H Poisson-commutes with an extra independent quadratic integral I, quadratic in the configuration space coordinates. Forgetting its physical interpretation and just considering it as a dynamical system, the possibility of finding generalizations of it was hopeless in view of a uniqueness theorem by Perelomov [4]. A close examination of the hypotheses under which this uniqueness result is obtained shows that some room is left for generalization if one does remain on $T^*\,S^2$. However the equations to be solved for these generalizations are somewhat involved and we were happy enough to get one using symbolic computation (Section 3). This example can be further generalized and gives rise to a countable family of integrable systems, which we show to be different from the family given earlier by Wojciechowski (Section 4). We conclude by proving that, using the so-called "minimal" quantization, our class of models are also integrable at the quantum level (Section 5).

2 Neumann integrable system

This integrable system is defined from the Lie algebra $\mathcal{G} = e(3)$ with respect to the Poisson bracket defined by

$$\{M_i, M_j\} = \epsilon_{ijk}\, M_k, \qquad \{M_i, X_j\} = \epsilon_{ijk}\, X_k, \qquad \{X_i, X_j\} = 0, \qquad i, j, k = 1, 2, 3. \tag{1}$$

The hamiltonian flow is given by

$$\dot{M}_i = \{H, M_i\}, \qquad \dot{X}_i = \{H, X_i\}. \tag{2}$$

It is easy to check that

$$C_1 = \sum_i X_i^2, \qquad \& \qquad C_2 = \sum_i X_i\, M_i$$

are two Casimir functions. Considering them as constants, for instance $C_1 = 1$ and $C_2 = 0$, we obtain the orbit of the co-adjoint representation of the group $G = E(3)$ which is the four-dimensional phase space $\Omega = T^*\,S^2$ of the considered system. Neumann hamiltonian [5] can be taken as

$$\left\{ \begin{array}{c} H = H^{(2)} + U, \\[2mm] H^{(2)} = \sum_i a_i\, M_i^2, \quad U = a_2 a_3\, X_1^2 + a_3 a_1\, X_2^2 + a_1 a_2\, X_3^2. \end{array} \right. \tag{3}$$

Its integrability follows from the existence of the extra conserved quantity

$$\left\{ \begin{array}{c} I = I^{(2)} + V, \\[2mm] I^{(2)} = \sum_i M_i^2, \quad V = (a_2 + a_3) X_1^2 + (a_3 + a_1) X_2^2 + (a_1 + a_2) X_3^2, \end{array} \right. \tag{4}$$

which Poisson-commutes with the Hamiltonian.

It is interesting to see how, given H, one can construct I. Taking into account that $\{M_i, f\} = -\hat{L}_i\, f \equiv -\epsilon_{ijk}\, X_j \partial_k\, f$ one has

$$\{H, I\} = \{H^{(2)}, V\} - \{I^{(2)}, U\} = 2 \sum_i M_i\, \widehat{L}_i \Big(U - a_i\, V \Big). \tag{5}$$

So if we write $V = a\, X_1^2 + b\, X_2^2 + c\, X_3^2$, the strict vanishing of the Poisson bracket requires

$$b - c = -a_1(a_2 - a_3), \qquad c - a = -a_2(a_3 - a_1), \qquad a - b = -a_3(a_1 - a_2). \tag{6}$$

Obviously these 3 relations add up to zero, so only two of them are independent and we get

$$V = a\, C_1 + (a_1 a_3 - a_2 a_3) X_2^2 + (a_1 a_2 - a_2 a_3) X_3^2, \tag{7}$$

which displays the *uniqueness* of V_1, up to the Casimir C_1. This uniqueness is stressed in proposition 1 of [4]. The choice $a = a_2 a_3$ gives then the Neumann potential (4).

3 A first Neumann-like integrable system

Our starting observation is simply that uniqueness is a result of the strong requirement of vanishing of the 3 terms appearing in (5). This is certainly sufficient to get uniquely Neumann system, but it is not necessary. We could have, rather

$$\{H, I\} = \cdots (C_1 - 1) + \cdots C_2$$

and this is still conserved in Ω. Under this weaker hypothesis we could hope for some Neumann-like integrable systems, with new potentials U_2 and V_2. Indeed the equations to be integrated become

$$\widehat{L}_i \Big(U_2 - a_i\, V_2 \Big) = \lambda_i (C_1 - 1) + \mu\, X_i, \tag{8}$$

where $(\lambda_i,\, \mu)$ are unknown functions of the X_i. These equations are quite difficult to integrate in general, so we have been looking for a specific example where U_2 and V_2 are quartic polynomials [1] and we have used Maple to solve for the equations. Quite surprisingly the solution, which is rather involved in the coordinates X_i, can be written in a rather simple form in terms of U and V:

$$U_2 = U\, V \qquad V_2 = V^2 - U. \tag{9}$$

Once this is observed, it is easy to give an analytic proof:

Proposition 1 *The dynamical system* $H_2 = H^{(2)} + U_2$ *and* $I_2 = I^{(2)} + V_2$, *with phase space* $T^*\, S^2$, *is integrable in Liouville sense.*

Proof: We start from

$$\widehat{L}_1(U_2 - a_1 V_2) = (U - a_1 V + a_1^2) \widehat{L}_1 V, \tag{10}$$

use the relation

$$U - a_1 V + a_1^2 = -(a_1 - a_2)(a_3 - a_1) X_1^2 + a_1^2 (1 - C_1),$$

[1] We were not able to find any solution with cubic polynomials.

and

$$\widehat{L}_1 V = -2(a_2 - a_3)X_2X_3, \qquad \widehat{L}_1 U = a_1\,\widehat{L}_1 V,$$

which lead to

$$\widehat{L}_1(U_2 - a_1V_2) = -2S(X)\,X_1 + 2a_1^2(a_2 - a_3)X_2X_3(1 - C_1), \tag{11}$$

with the totally symmetric function

$$S(X) = (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)X_1X_2X_3. \tag{12}$$

Summing the various terms in (5) we get

$$\begin{aligned} &\{H_2, I_2\} = -4S(X)\,C_2 + \\ &+4(1 - C_1)\left[a_1^2(a_2 - a_3)X_2X_3M_1 + a_2^2(a_3 - a_1)X_3X_1M_2 + a_3^2(a_1 - a_2)X_1X_2M_3\right], \end{aligned} \tag{13}$$

which vanishes in Ω. So this system is integrable . $\square$

4 More Neumann-like integrable systems

The previous result can be generalized to polynomials of even degree in the following way. Let us define the series U_n and V_n by the recurrence:

$$\begin{cases} U_1 = U \\ V_1 = V \end{cases} \qquad \begin{cases} U_n = U\,V_{n-1} \\ V_n = V\,V_{n-1} - U_{n-1} \end{cases} \qquad n \geq 2. \tag{14}$$

Standard techniques give the following useful information on these polynomials:

Proposition 2 *The explicit form of the polynomials is*

$$\begin{cases} U_n = \displaystyle\sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-1-k}{k} U^{k+1}V^{n-1-2k}, \\ V_n = \displaystyle\sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} U^k V^{n-2k}, \end{cases} \tag{15}$$

and they verify the following partial differential equations:

$$\partial_V U_n + U\partial_U V_n = 0, \qquad \partial_V V_n + V\partial_U V_n = \partial_U U_n, \qquad n \geq 1. \tag{16}$$

Proof:
We first need to prove the three terms recurrence relation

$$V_{n+1} - VV_n + UV_{n-1} = 0, \qquad n \geq 2.$$

Using the following identity for the binomial coefficients

$$\binom{n+1-k}{k} - \binom{n-k}{k} = \binom{n-k}{k-1}(1 - \delta_{k0}), \quad k \geq 0,$$

and the explicit form of V_n one gets

$$V_{n+1} - VV_n = -\sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-1-k}{k} U^{k+1} V^{n-1-2k} = -UV_{n-1}.$$

The first partial differential equation follows from the identity

$$k\binom{n-k}{k} = (n+1-2k)\binom{n-k}{k-1}, \qquad k \geq 1,$$

and the second one from

$$(n-2k)\binom{n-k}{k} - (k+1)\binom{n-1-k}{k} = (k+1)\binom{n-1-k}{k+1}, \qquad k \geq 0.$$

In the analysis some care is required with the upper bounds of the summations. $\square$
We are now in position to prove:

Proposition 3 *The dynamical systems $H_n = H^{(2)} + U_n$ and $I_n = I^{(2)} + V_n$, with phase space $T^*\,S^2$, are integrable in Liouville sense.*

Proof:
Using the recurrence relations for the polynomials U_n and V_n we have first

$$\widehat{L}_1(U_n - a_1 V_n) = (U - a_1 V)\widehat{L}_1 V_{n-1} + a_1 \widehat{L}_1 U_{n-1}.$$

The generic relation

$$\widehat{L}_1\, f = \Big(\partial_V f + a_1 \partial_U f\Big)\, \widehat{L}_1 V,$$

used in the previous equation gives for the right hand side

$$(U - a_1 V)\partial_V V_{n-1} + a_1(U\partial_U V_{n-1} + \partial_V U_{n-1}) + a_1^2(\partial_U U_{n-1} - V\partial_U V_{n-1}).$$

The partial differential equations of proposition 2 give then

$$\widehat{L}_1(U_n - a_1 V_n) = \partial_V V_{n-1}(U - a_1 V + a_1^2)\widehat{L}_1 V = \partial_V V_{n-1}\, \widehat{L}_1(U_2 - a_1 V_2), \tag{17}$$

and since $\partial_V V_{n-1}$ is fully symmetric we get

$$\{H_n, I_n\} = \partial_V V_{n-1}\{H_2, I_2\}, \tag{18}$$

which proves the proposition. $\square$

Let us show that the family of potentials obtained here is indeed different from a family obtained by Wojciechowski in [7]. This author has obtained a countable set of integrable potentials on $T^*\,S^n$ which we have to restrict to $n = 2$. There is no general formula for his potentials, but the simplest ones are given page 109 , which we will reproduce (we take of course $r = 1$). The first three are [2]

$$\begin{aligned} I &= \sum_k a_k\, X_k^2, && (19)\\ I_2 &= \sum_k a_k^2\, X_k^2 - (\sum a_k\, X_k^2)^2, && (20)\\ I_3 &= \sum_k a_k^3\, X_k^2 - 2(\sum_j a_j\, X_j^2)(\sum_k a_k^2\, X_k^2) + (\sum_k a_k\, X_k^2)^3. && (21) \end{aligned}$$

[2] We discard Braden's potential $(\sum_k X_k^2/a_k)^{-1}$ since it is not a polynomial.

Let us compare our potentials with these ones, beginning with V. Using the constraint

$$C_1 \equiv X_1^2 + X_2^2 + X_3^2 = 1, \tag{22}$$

we can write

$$V = (a_2 + a_3)X_1^2 + (a_3 + a_1)X_2^2 + (a_1 + a_2)X_3^2 = \sum_k a_k - I$$

and, up to a constant, it coincides with the potential (19) given by W. This corresponds to Neumann on $T^* S^2$.

Then our potential V_2 is given by

$$V_2 = -U + V^2 = -(a_2 a_3\, X_1^2 + a_3 a_1\, X_2^2 + a_1 a_2\, X_3^2) + (\sum_k a_k - \sum_k a_k\, X_k^2)^2,$$

to be compared with the quartic potential I_2. One can check the relation

$$V_2 + I_2 = (a_2 a_3 - a_1^2)X_1^2 + (a_3 a_1 - a_2^2)X_2^2 + (a_1 a_2 - a_3^2)X_3^2 + a_1^2 + a_2^2 + a_3^2,$$

which shows that the quartic terms in V_2 and I_2 are just opposite in sign but that their quadratic content is *different* even using the constraint (22). So our potential V_2 is definitely different from the potential I_2 given by Wojciechowski.

Let us now consider our potential V_3 against I_3. From the recurrence given in our article we have

$$\begin{aligned} V_3 &= -2UV + V^3 = \\ &= -2(\textstyle\sum_k a_k - \sum_k a_k\, X_k^2)(a_2 a_3\, X_1^2 + a_3 a_1\, X_2^2 + a_1 a_2\, X_3^2) + (\sum_k a_k - \sum_k a_k\, X_k^2)^3, \end{aligned}$$

and upon expanding, using the constraint and with some algebra we get

$$\begin{aligned} V_3 + I_3 = {} & (a_1 - a_2)(a_1 - a_3)(3a_1 + 2a_2 + 2a_3)\, X_1^4 + \text{ circ. perm.} \\ & + a_1(-4a_1^2 + 3a_2^2 + 3a_3^2)\, X_1^2 + \text{ circ. perm.} \\ & + a_1^2(a_1 - a_2 - a_3) + \text{ circ. perm. } + a_1 a_2 a_3. \end{aligned} \tag{23}$$

Here the sextic terms in V_3 and I_3 are again oposite in sign but their quartic terms are *different*. Let us recall that the quartic terms in $-V_2$ and I_2 (which are equal) were

$$(a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2)^2$$

so we cannot express the quartic terms in (23) using such a term. So our integrable potential V_3 is indeed different from the potential I_3 of Wojciechowski.

It would be quite cumbersome to give a general comparison of the results for the countable set of potentials, but we hope that these arguments are sufficient to show that our integrable potentials are different from the ones considered by Wojciechowski.

5 Quantization

Let us discuss briefly the quantization of these models. Since there are no quantization ambiguities we do not expect any problem with quantum integrability. The quantum observables should verify

$$[\widehat{M}_i, \widehat{M}_j] = -i\epsilon_{ijk}\,\widehat{M}_k, \qquad [\widehat{M}_i, \widehat{X}_j] = -i\epsilon_{ijk}\,\widehat{X}_k, \qquad [\widehat{X}_i, \widehat{X}_j] = 0, \qquad i,j,k = 1,2,3. \tag{24}$$

For notational convenience we will use also $\widehat{M}_i \equiv \mathcal{Q}(M_i)$, etc... Then the classical quantities are unambiguously quantized as

$$\widehat{H}_n \equiv \mathcal{Q}(H_n) = \sum_i a_i \widehat{M}_i^2 + \widehat{U}_n, \qquad \widehat{I}_n \equiv \mathcal{Q}(I_n) = \sum_i \widehat{M}_i^2 + \widehat{V}_n, \tag{25}$$

and the constraints are now operator valued:

$$\widehat{C}_1 = \sum_i \widehat{X}_i^2 = Id, \qquad \widehat{C}_2 = \sum_i \widehat{X}_i \widehat{M}_i = \sum_i \widehat{M}_i \widehat{X}_i = 0. \tag{26}$$

To prove the quantum conservation we start from

$$[\widehat{H}_n, \widehat{I}_n] = [\widehat{H}^{(2)}, \widehat{V}_n] - [\widehat{I}^{(2)}, \widehat{U}_n] \tag{27}$$

which gives

$$[\widehat{H}_n, \widehat{I}_n] = \sum_i \widehat{M}_i\,[\widehat{M}_i, a_i\widehat{V}_n - \widehat{U}_n] + \sum_i [\widehat{M}_i, a_i\widehat{V}_n - \widehat{U}_n]\,\widehat{M}_i. \tag{28}$$

One can check that

$$[\widehat{M}_i, a_i\widehat{V}_n - \widehat{U}_n] = -i\mathcal{Q}(\{M_i, a_iV_n - U_n\}) = -i\mathcal{Q}(\widehat{L}_i(U_n - a_iV_n)). \tag{29}$$

We have seen in (11) that

$$\widehat{L}_1(U_n - a_1V_n) = \partial_V V_{n-1}\Big(- 2S(X)X_1 + 2a_1^2(a_2 - a_3)(1 - C_1(X))X_2X_3\Big), \tag{30}$$

and since only X-dependence is involved one has

$$\begin{aligned} \mathcal{Q}(\widehat{L}_1(U_n - a_1V_n)) &= \widehat{\partial_V V_{n-1}}\Big(- 2\widehat{S(X)}\widehat{X}_1 + 2a_1^2(a_2 - a_3)\widehat{X}_2\widehat{X}_3(Id - \widehat{C}_1)\Big) \\ &= -2\widehat{\partial_V V_{n-1}}\,\widehat{S(X)}\widehat{X}_1. \end{aligned} \tag{31}$$

and then replacing this result into (28) we end up with

$$[\widehat{H}_n, \widehat{I}_n] = -2\left(\sum_i \widehat{M}_i\widehat{X}_i\right)\widehat{S(X)}\widehat{\partial_V V_{n-1}} - 2\widehat{\partial_V V_{n-1}}\widehat{S(X)}\left(\sum_i \widehat{X}_i\widehat{M}_i\right) = 0.$$

This argument is quite rough: it would be more complete if one were able to use true coordinates in $T^*\,S^2$ and to quantize the unconstrained theory, using for instance the “minimal” quantization scheme as developed in [2]. Notice that the Lax pair is known for the Neumann system [6], but not for these Neumann-like systems. This lacking piece of knowledge could possibly be of great help in finding the separation variables and handling the unconstrained quantization problem mentioned above.

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